MATHEMATICS Proceedings A 87 (2), June 18, 1984

A theory of generalized functions based on holomorphic semi-groups

Part B: Analyticity spaces, trajectory spaces and their pairing

by J. de Graaf

Technological University, Department of Mathematics, Eindhoven, the Netherlands

Communicated by Prof. N.G. de Bruijn at the meeting of September 26, 1983

CHAPTER 1. THE ANALYTICITY SPACE $S_{X,\mathscr{A}}$

In a complex separable Hilbert space with inner product $(\cdot,\cdot)_X$ we consider an unbounded non-negative self-adjoint operator $\mathscr{A}$. This operator will be fixed throughout chapters 1, 2, 3. Since $\mathscr{A}$ is self-adjoint it admits a spectral resolution

$$\mathscr{A}=\int_0^\infty \lambda\, d\mathscr{E}_\lambda .$$

For details on such spectral resolutions, see [Y], [LS].

By

$$\int_a^b \psi(\lambda)\, d\mathscr{E}_\lambda,\ 0\le a<b\le\infty,$$

we mean

$$\int_{-\infty}^{\infty} \chi_{a,b}(\lambda)\psi(\lambda)\, d\mathscr{E}_\lambda$$

with

$$\chi_{a,b}(\lambda)=\begin{cases} 1 \text{ on } (-1,b] \text{ if } a=0\\ 1 \text{ on } (a,b] \text{ if } a>0\\ 0 \text{ elsewhere.}\end{cases}$$

we define

$$e^{-t\mathscr{A}} = \int_0^\infty e^{-\lambda t} d\mathscr{E}_\lambda, \ t \in \mathbb{C}.$$

For $t \in \mathbb{R}$ the operator $e^{-t\mathscr{A}}$ is self-adjoint, $e^{-t\mathscr{A}}$ is bounded iff Re $t \geq 0$. $e^{-t\mathscr{A}}$ is unitary iff Re $t=0$. Further $e^{-t\mathscr{A}}$ is invertible and $\forall t \in \mathbb{C}\ (e^{-t\mathscr{A}})^{-1} = e^{t\mathscr{A}}$.

On $\mathbb{C}^+$, i.e. the set of $t \in \mathbb{C}$ with Re $t>0$, the operators $e^{-t\mathscr{A}}$ establish a one-parameter semi-group of bounded injective operators on $\boldsymbol{X}$.

We now introduce our test space $\boldsymbol{S}_{\mathrm{X},\mathscr{A}}$. This space equals the set of analytic vectors for the operator $\mathscr{A}$. Cf. [Ne].

DEFINITION 1.1.

$$\boldsymbol{S}_{\mathrm{X},\mathscr{A}} = \bigcup_{\mathrm{Re}\, t>0} \{e^{-t\mathscr{A}}x \mid x \in \boldsymbol{X}\} = \bigcup_{\mathrm{Re}\, t>0} e^{-t\mathscr{A}}(\boldsymbol{X}).$$

$\boldsymbol{S}_{\mathrm{X},\mathscr{A}}$ is a dense linear subspace of $\boldsymbol{X}$. From the semi-group properties the following equalities immediately follow for any $\delta>0$

$$\boldsymbol{S}_{\mathrm{X},\mathscr{A}} = \bigcup_{t>0} e^{-t\mathscr{A}}(\boldsymbol{X}) = \bigcup_{0<t<\delta} e^{-t\mathscr{A}}(\boldsymbol{X}) = \bigcup_{n\in\mathbb{N},\, 0<(1/n)<\delta} e^{-(1/n)\mathscr{A}}(\boldsymbol{X}).$$

Each of the spaces $e^{-t\mathscr{A}}(\boldsymbol{X})$, $t>0$, can be considered as a Hilbert space. The inner product is

$$(u,v)_t = (e^{t\mathscr{A}}u, e^{t\mathscr{A}}v)_{\boldsymbol{X}} = \int_0^\infty e^{2\lambda t} d(\mathscr{E}_\lambda u, v).$$

The completeness of $e^{-t\mathscr{A}}(\boldsymbol{X})$ follows from the closedness of $e^{t\mathscr{A}}$. $e^{-t\mathscr{A}}(\boldsymbol{X})$ consists of exactly those $u \in \boldsymbol{X}$ for which

$$\int_0^\infty e^{2\lambda t} d(\mathscr{E}_\lambda u, u) < \infty.$$

If locally no confusion is likely to arise we suppress as many subscripts as possible. In chapters 1, 2 and 3 we shall write consistently: $\boldsymbol{S}$ for $\boldsymbol{S}_{\mathrm{X},\mathscr{A}}$, $\boldsymbol{X}_t$ for $e^{-t\mathscr{A}}(\boldsymbol{X})$, $(\cdot,\cdot)$ for $(\cdot,\cdot)_{\boldsymbol{X}}$.

DEFINITION 1.2. *The strong topology on* $\boldsymbol{S}$ *is the finest locally convex topology on* $\boldsymbol{S}$ *for which the natural injections* $i_t : \boldsymbol{X}_t \to \boldsymbol{S}$, $t>0$, *are all continuous.*

In other words: $\boldsymbol{S}$ is made into a locally convex topological vector space by imposing the inductive limit topology with respect to the family $\{\boldsymbol{X}_t\}_{t>0}$. Cf. [SCH] Ch. II.6. Since the natural injections $\boldsymbol{X}_\tau \to \boldsymbol{X}_t$, $\tau>t>0$, are all continuous the inductive limit topology is already brought about by the family $\{\boldsymbol{X}_{1/n}\}_{n\in\mathbb{N}}$. A subset $\mathfrak{U} \subset \boldsymbol{S}$ is open iff for every $t>0$ the set $i_t^{-1}(\mathfrak{U}) = \mathfrak{U} \cap \boldsymbol{X}_t$ is open in $\boldsymbol{X}$ or, equivalently, iff for every $n \in \mathbb{N}$ the set $\mathfrak{U} \cap \boldsymbol{X}_{1/n}$ is open in $\boldsymbol{X}_{1/n}$. The inductive limit here is more complicated than in the case of the Schwartz test function spaces because our inductive limit is not strict! A closed subset of

$\boldsymbol{X}_\tau$ when considered as a set in $\boldsymbol{X}_t$, $0<t<\tau$, is not necessarily closed. We shall see that open sets in $\boldsymbol{S}$ are always unbounded.

We introduce some notations:

- $\mathbb{B}$ denotes the set of everywhere finite real valued Borel functions on $\mathbb{R}$ such that $\forall t>0$ the function $\psi(x)e^{-tx}$ is bounded on $[0,\infty)$.
- $\mathbb{B}_+ \subset \mathbb{B}$ contains those ψ for which there exists $\varepsilon>0$ such that $\psi(x)\geq\varepsilon>0$, for all $x\geq 0$.

Let $\psi\in\mathbb{B}$. Then $\psi(\mathscr{A})$ is defined by

$$\psi(\mathscr{A})=\int_0^\infty \psi(\lambda)d\mathscr{E}_\lambda.$$

The domain of $\psi(\mathscr{A})$, denoted by $\mathscr{D}(\psi(\mathscr{A}))$, consists of exactly those $x\in\boldsymbol{X}$ with

$$\int_0^\infty \psi^2(\lambda)d(\mathscr{E}_\lambda x, x)<\infty.$$

It follows that $\boldsymbol{S}$ is contained in the domain of each operator $\psi(\mathscr{A})$.

Further: $\forall\psi\in\mathbb{B}$, $\forall t>0$ the operators $\psi(\mathscr{A})e^{-t\mathscr{A}}$ are bounded and self-adjoint. The sets of operators corresponding to the sets $\mathbb{B}$ and $\mathbb{B}_+$ will be denoted by $\mathbb{B}(\mathscr{A})$ and $\mathbb{B}_+(\mathscr{A})$, respectively. The operators in $\mathbb{B}_+(\mathscr{A})$ are all strictly positive.

DEFINITION 1.3. *For each $\psi\in\mathbb{B}_+$ we introduce the (semi-) norm $\mathfrak{p}_\psi$ by*

$$\mathfrak{p}_\psi(u)=\|\psi(\mathscr{A})u\|=\{\int_0^\infty \psi(\lambda)^2 d(\mathscr{E}_\lambda u,u)\}^{\frac{1}{2}},\ u\in\boldsymbol{S}.$$

Further for $\psi\in\mathbb{B}_+$ and $\varepsilon>0$ we define the set

$$\mathfrak{U}_{\psi,\varepsilon}=\{u\in\boldsymbol{S}\,|\,\|\psi(\mathscr{A})u\|<\varepsilon\}.$$

The next theorem is very fundamental. It tells that the strong topology in $\boldsymbol{S}_{\mathrm{X},\mathscr{A}}$ is generated by the semi-norms $\mathfrak{p}_\psi$.

THEOREM 1.4.

I. *$\forall\psi\in\mathbb{B}_+$, $\forall\varepsilon>0$ $\mathfrak{U}_{\psi,\varepsilon}$ is a convex, balanced, absorbing open set in the strong topology. In other words the semi-norms $\mathfrak{p}_\psi$ are continuous.*

II. *Let a convex set $\Omega\subset\boldsymbol{S}$ be such that for each $t>0$ $\Omega\cap\boldsymbol{X}_t$ contains a neighbourhood of 0 in $\boldsymbol{X}_t$. Then Ω contains a set $\mathfrak{U}_{\psi,\varepsilon}$ with $\psi\in\mathbb{B}_+$ and $\varepsilon>0$.*

PROOF. I. A standard inner product argument shows that $\mathfrak{U}_{\psi,\varepsilon}$ is convex, balanced and absorbing. For the terminology see [Y] Ch. I.1. We now show that for each $t>0$ $\mathfrak{U}_{\psi,\varepsilon}\cap\boldsymbol{X}_t$ is an open set in $\boldsymbol{X}_t$.

By definition

$$\mathfrak{U}_{\psi,\varepsilon}\cap\boldsymbol{X}_t=\{u|u\in\boldsymbol{X}_t,\ \|\psi(\mathscr{A})u\|<\varepsilon\}.$$

Because of $\|\psi(\mathscr{A})u\| \leq \|\psi(\mathscr{A})e^{-t\mathscr{A}}\| \, \|e^{t\mathscr{A}}u\|$ and the boundedness of $\psi(\mathscr{A})e^{-t\mathscr{A}}$, the norm $\mathbf{p}_\psi$ is continuous on $\boldsymbol{X}_t$.

II. Introduce the operator $\mathscr{P}_n = \int_{n-1}^{n} d\mathscr{E}_\lambda$, $n \in \mathbb{N}$. Let r_n be the radius of the largest open ball in $\mathscr{P}_n(\boldsymbol{X})$ which fits in $\Omega \cap \mathscr{P}_n(\boldsymbol{X})$. So

$$r_n = \sup \{\varrho | [u \in \mathscr{P}_n(\boldsymbol{X}) \wedge \|\mathscr{P}_n u\| < \varrho] \Rightarrow u \in \mathscr{P}_n(\Omega)\}.$$

Next define $\chi : \mathbb{R} \to \mathbb{R}$ as follows

$$\chi(\lambda) = \begin{cases} 0 & \text{if } \lambda < 0 \\ 2 \max\left(\dfrac{n^2}{r_n}, 1\right) & \text{if } \lambda \in (n-1, n],\ n \in \mathbb{N} \\ \chi(\frac{1}{2}) & \text{if } \lambda = 0. \end{cases}$$

We prove $\chi \in \mathbb{B}_+$. Let $t > 0$. Then there is $\varepsilon > 0$ such that

$$\{u | \int_0^\infty e^{\lambda t} d(\mathscr{E}_\lambda u, u) < \varepsilon^2\} \subset \Omega \cap \boldsymbol{X}_{\frac{1}{2}t}$$

because $\Omega \cap \boldsymbol{X}_{\frac{1}{2}t}$ contains an open neighbourhood of 0.

Thus we find that for all $n \in \mathbb{N}$, $r_n > \varepsilon e^{-\frac{1}{2}nt}$. Hence for $\lambda \in (n-1, n]$

$$\chi(\lambda)e^{-\lambda t} < 2e^{-(n-1)t} \max\left(\frac{n^2}{r_n}, 1\right) \leq 2\left(\frac{n^2}{\varepsilon} e^{-\frac{1}{2}nt} + 1\right)e^t.$$

So

$$\sup_{\lambda \geq 0} e^{-\lambda t}\chi(\lambda) < \infty.$$

We now show

$$(*) \qquad \|\chi(\mathscr{A})u\| < 1 \Rightarrow u \in \Omega.$$

Suppose $u \in \boldsymbol{X}_t$ for some $t > 0$. Then $\sum_{n=1}^\infty \|\mathscr{P}_n u\|_t^2 < \infty$ and for some τ, $0 < \tau < t$

$$(\underset{*}{*}) \qquad \|\mathscr{P}_n u\|_\tau^2 \leq e^{-2(n-1)(t-\tau)} \|u\|_t^2.$$

Further, because of our assumption (*)

$$\|\mathscr{P}_n u\| \leq \tfrac{1}{2} \min(n^{-2} r_n, 1).$$

Therefore $2n^2 \mathscr{P}_n u \in \Omega \cap \boldsymbol{X}_\tau$ for every $n \in \mathbb{N}$.

In $\boldsymbol{X}_\tau$ we represent u by

$$u = \sum_{n=1}^N \frac{1}{2n^2} (2n^2 \mathscr{P}_n u) + \left(\sum_{n=N}^\infty \frac{1}{2n^2}\right) u_N$$

with

$$u_N = \left(\sum_{j=N}^\infty \frac{1}{2j^2}\right)^{-1} \sum_{n=N}^\infty \mathscr{P}_n u.$$

With (*) we calculate

$$\|u_N\|_\tau^2 \leq 4N^4 \sum_{n=N}^{\infty} \|\mathscr{P}_n u\|_\tau^2 \leq 4N^4 e^{(-2N+2)(t-\tau)} \|u\|_t^2.$$

Since $\Omega \cap \boldsymbol{X}_\tau$ contains an open neighbourhood of 0 for N sufficiently large $u_N \in \Omega \cap \boldsymbol{X}_\tau$.

Finally we gather that u is a sub-convex combination of elements in $\Omega \cap \boldsymbol{X}_\tau$ which is a convex set. A posteriori it is clear that $u \in \Omega \cap \boldsymbol{X}_t$. □

REMARK. Similar to the proof of part I one proves that the sets

$$\bar{\mathfrak{U}}_{\psi,\varepsilon} = \{u \in \boldsymbol{S} \,|\, \|\psi(\mathscr{A})u\| \cap \varepsilon\}, \ \psi \in \mathbb{B}_+, \ \varepsilon > 0,$$

are closed.

DEFINITION 1.5. *A subset $\mathfrak{W} \subset \boldsymbol{S}$ is called bounded if for each 0-neighbourhood $\mathfrak{U}$ in $\boldsymbol{S}$ there exists a complex number λ such that $\mathfrak{W} \subset \lambda\mathfrak{U}$. Cf.* [SCH] 1.5.

The next theorem characterizes bounded sets in $\boldsymbol{S}$.

THEOREM 1.6. *A subset $\mathfrak{W} \subset \boldsymbol{S}$ is bounded iff*

$$\exists t > 0 \ \mathfrak{W} \subset \boldsymbol{X}_t$$

and

$$\exists M > 0 \ \forall u \in \mathfrak{W} \ \|u\|_t \leq M.$$

PROOF. $\Leftarrow$) Let $\psi \in \mathbb{B}_+$ and $u \in \mathfrak{W}$. Then

$$\|\psi(\mathscr{A})u\| \leq \|\psi(\mathscr{A})e^{-t\mathscr{A}}\| \, \|e^{t\mathscr{A}}u\| \leq M \|\psi(\mathscr{A})e^{-t\mathscr{A}}\|.$$

$\Rightarrow$) Taking $\psi = 1$ we observe that $\mathfrak{W}$ is a bounded set in $\boldsymbol{X}$. Denote its bound by ϱ. Suppose the statement were not true, then

$$\forall t > 0 \ \forall M > 0 \ \exists u \in \mathfrak{W} \|e^{t\mathscr{A}}u\| > M.$$

By induction we define two sequences of real numbers $\{t_n\}$, $\{N_n\}$, $t_n \downarrow 0$, $N_n \uparrow \infty$ as $n \to \infty$ and a sequence $\{u_n\} \subset \mathfrak{W}$ as follows:

$n=1$: Choose $t_1 > 0$ and $M=2$. Then take $N_1 > 0$ and $u_1 \in \mathfrak{W}$ such that

$$\int_0^{N_1} e^{2t_1\lambda} d(\mathscr{E}_\lambda u_1, u_1) > 1.$$

$n = l+1$: Suppose

$$\forall t \leq \tfrac{1}{2} t_l \ \forall u \in \mathfrak{W} \ \forall K > 0 \int_{N_l}^{N_l+K} e^{2t\lambda} d(\mathscr{E}_\lambda u, u) \leq l+1$$

is true. Then $\mathfrak{B}$ is a bounded set in $\boldsymbol{X}_t$, $t \leq \frac{1}{2}t_l$, because

$$\int_0^\infty e^{2\lambda t} d(\mathscr{E}_\lambda u, u) = \int_0^{N_l} + \int_{N_l}^\infty \leq \varrho^2 e^{2tN_l} + l + 1.$$

If not, choose $t_{l+1} \leq \frac{1}{2}t_l$, $u_{l+1} \in \mathfrak{B}$ and N_{l+1} such that

$$\int_{N_l}^{N_{l+1}} e^{2t_{l+1}\lambda} d(\mathscr{E}_\lambda u_{l+1}, u_{l+1}) > l + 1.$$

If our sequence terminates for a certain value of n then $\mathfrak{B}$ is a bounded set. If not, define the function $\eta(\lambda)$ on $(0, \infty)$ by

$$\eta(\lambda) = e^{\lambda t_n} \text{ on the interval } [N_{n-1}, N_n],\ n = 1, 2, \ldots.$$

Since $\eta(\lambda)e^{-\lambda t}$ is a bounded function for $t > 0$ we have $\eta \in \mathbb{B}_+$. Then because of the assumption the sequence $\eta(\mathscr{A})u_n$ should be bounded. However

$$\|\eta(\mathscr{A})u_n\|^2 = \int_0^\infty \eta^2(\lambda) d(\mathscr{E}_\lambda u_n, u_n) > n + 1.$$

Contradiction! □

In the next theorem we give a characterization for the convergence of sequences $\{u_n\}$ in the strong topology of $\boldsymbol{S}$.

THEOREM 1.7. *$u_n \to 0$ in the strong topology of $\boldsymbol{S}$ iff*

$$\exists t > 0\ \{u_n\} \subset \boldsymbol{X}_t \text{ and } \|u_n\|_t \to 0.$$

PROOF. $\Leftarrow$) For any $\psi \in \mathbb{B}_+$ we have $\|\psi(\mathscr{A})u_n\| \leq \|\psi(\mathscr{A})e^{-t\mathscr{A}}\|\ \|e^{t\mathscr{A}}u_n\| \to 0$ as $n \to \infty$ because $\psi(\mathscr{A})e^{-t\mathscr{A}}$ is a bounded operator on $\boldsymbol{X}$.

$\Rightarrow$) Suppose $u_n \to 0$. Then $\forall \psi \in \mathbb{B}_+\ \|\psi(\mathscr{A})u_n\| \to 0$.

We conclude that the sequence $\{u_n\}$ is a bounded set. So $\exists \theta > 0\ \forall n \in \mathbb{N}$ $\|u_n\|_\tau \leq \theta$. Further, taking $\psi = 1$, it is clear that $\|u_n\|_0 \to 0$ as $n \to \infty$. From this we derive

$$(*) \qquad \forall \alpha > 0\ \forall t > 0 \int_0^\alpha e^{2\lambda t} d(\mathscr{E}_\lambda u_n, u_n) \to 0 \text{ as } n \to \infty.$$

Now we show $\|u_n\|_t \to 0$ for any $t < \tau$.

$$(\substack{*\\*}) \qquad \|u_n\|_t^2 = \int_0^L e^{2\lambda t} d(\mathscr{E}_\lambda u_n, u_n) + \int_L^\infty e^{2\lambda t} d(\mathscr{E}_\lambda u_n, u_n).$$

The second integral can be estimated uniformly

$$\int_L^\infty e^{2\lambda t} d(\mathscr{E}_\lambda u_n, u_n) = \int_L^\infty e^{-2\lambda(\tau - t)} e^{2\lambda\tau} d(\mathscr{E}_\lambda u_n, u_n) \leq$$

$$\leq e^{-2L(\tau - t)} \int_L^\infty e^{2\lambda\tau} d(\mathscr{E}_\lambda u_n, u_n) \leq e^{-2L(\tau - t)}\theta^2.$$

By taking L sufficiently large the second term in ($\overset{*}{*}$) can be made smaller than $\frac{1}{2}\varepsilon$ uniformly in n.

The first term in ($\overset{*}{*}$) tends to 0 as $n\to\infty$ because of (*). □

THEOREM 1.8.

I. *Suppose* $\{u_n\}$ *is a Cauchy sequence in the strong topology of* $\boldsymbol{S}$. *Then* $\exists t>0$ $\{u_n\}\subset\boldsymbol{X}_t$ *and* $\{u_n\}$ *is a Cauchy sequence in* $\boldsymbol{X}_t$.

II. $\boldsymbol{S}$ *is sequentially complete, i.e. every Cauchy sequence converges to a limit point.*

PROOF. I. An argument similar to the proof of the preceding section.

II. Follows from I and the completeness of $\boldsymbol{X}_t$. □

Next we characterize compact sets in $\boldsymbol{S}$.

THEOREM 1.9. *A subset* $\mathfrak{K}\subset\boldsymbol{S}$ *is compact iff*

$$\exists t>0\ \mathfrak{K}\subset\boldsymbol{X}_t \text{ and } \mathfrak{K} \text{ is compact in } \boldsymbol{X}_t.$$

PROOF. $\Leftarrow$) Let $\{\Omega_\alpha\}$ be an open covering of $\mathfrak{K}$ in $\boldsymbol{S}$. Then $\{\Omega_\alpha\cap\boldsymbol{X}_t\}$ is an open covering of $\mathfrak{K}$ in $\boldsymbol{X}_t$. Since $\mathfrak{K}$ is supposed to be compact in $\boldsymbol{X}_t$ there exists a finite subcovering $\{\Omega_{\alpha_i}\}$, $1\le i\le N$, with $\bigcup_{i=1}^N(\Omega_{\alpha_i}\cap\boldsymbol{X}_t)\supset\mathfrak{K}$. But then certainly $\bigcup_{i=1}^N\Omega_{\alpha_i}\supset\mathfrak{K}$.

$\Rightarrow$) Since $\mathfrak{K}$ is compact it is bounded and therefore it is a bounded set with bound θ in $\boldsymbol{X}_\tau$ for some $\tau>0$. We show that $\mathfrak{K}$ is compact in $\boldsymbol{X}_t$ whenever $t<\tau$. Consider a sequence $\{u_n\}\subset\mathfrak{K}$. There exists a converging subsequence $\{u_{n_j}\}$ with $u_{n_j}\to u\in\mathfrak{K}$ as $j\to\infty$, convergence in $\boldsymbol{S}$. Then also $u_{n_j}\to u$ in $\boldsymbol{X}$-sense. Put $u_{n_j}-u=v_j$. $\{v_j\}$ is a bounded sequence in $\boldsymbol{X}_\tau$ and $\|v_j\|\to0$ as $j\to\infty$. Now we find ourselves in exactly the same position as in the proof of the only-if-part of Theorem 1.7. We conclude $\|v_j\|_t\to0$ whenever $0\le t<\tau$. Our subsequence $\{u_{n_j}\}$ converges to u in $\boldsymbol{X}_t$-sense. This shows the compactness of $\mathfrak{K}$ in $\boldsymbol{X}_t$ for $0\le t<\tau$. □

The next theorem gives an alternative description of the space $\boldsymbol{S}_{\mathrm{X},\mathscr{A}}$. The simple proof is omitted.

THEOREM 1.10. *Let* $u\in\boldsymbol{X}$ *be such that* $u\in\mathscr{D}(\psi(\mathscr{A}))$ *for all* $\psi\in\mathbb{B}_+$. *Then* $u\in\boldsymbol{S}$. *In other words*

$$\boldsymbol{S}_{\mathrm{X},\mathscr{A}}=\bigcap_{\psi\in\mathbb{B}_+}\mathscr{D}(\psi(\mathscr{A})).$$ □

In order to make a link with the literature on topological vector spaces we now describe the properties of our space $\boldsymbol{S}$ by using the standard terminology of topological vector spaces. [SCH].

The terminology is explained in the proof.

THEOREM 1.11.

I. $S_{X,\mathscr{A}}$ *is complete.*

II. $S_{X,\mathscr{A}}$ *is bornological.*

III. $S_{X,\mathscr{A}}$ *is barreled.*

IV. $S_{X,\mathscr{A}}$ *is Montel iff for every* $t>0$ *the operator* $e^{-t\mathscr{A}}$ *is compact as an operator on* $\boldsymbol{X}$.

V. $S_{X,\mathscr{A}}$ *is nuclear iff for every* $t>0$ *the operator* $e^{-t\mathscr{A}}$ *is a HS (=Hilbert-Schmidt) operator on* $\boldsymbol{X}$.

PROOF. I. Let $\{x_\alpha\}$ be a Cauchy net. The α's belong to a directed set D. For each neighbourhood $\Omega \ni 0$ there is $\gamma \in D$ such that whenever $\alpha > \gamma$ and $\beta > \gamma$ one has $x_\alpha - x_\beta \in \Omega$. We now prove that there exists an $x \in \boldsymbol{S}$ such that $x_\alpha \to x$ in the strong topology. Let x be the limit of x_α in $\boldsymbol{X}$-sense. For each $\psi \in \mathbb{B}_+$ the net $\{\psi(\mathscr{A})x_\alpha\}$ converges in $\boldsymbol{X}$-sense to a limit x_ψ. Because of the closedness of $\psi(\mathscr{A})$ one has $x_\psi \in \mathscr{D}(\psi(\mathscr{A}))$ and $x_\psi = \psi(\mathscr{A})x$. The result follows by applying Theorem 1.10.

II. Every circled convex subset $\Omega \subset \boldsymbol{S}$ that absorbs every bounded subset $\mathfrak{W} \subset \boldsymbol{S}$ has to be a neighbourhood of 0. Let $\mathfrak{B}_t$ be the open unit ball in $\boldsymbol{X}_t$, $t>0$. $\mathfrak{B}_t$ is bounded in $\boldsymbol{S}$, therefore for some $\varepsilon>0$ one has $\varepsilon\mathfrak{B}_t \subset \Omega \cap \boldsymbol{X}_t$.

We conclude that for every $t>0$ the set $\Omega \cap \boldsymbol{X}_t$ contains an open neighbourhood of 0. But then according to Theorem 1.4.II Ω contains a set $\mathfrak{U}_{\psi,\varepsilon}$.

III. A barrel $\mathfrak{B}$ is a subset which is radial, convex, circled and closed. We have to prove that every barrel contains an open neighbourhood of the origin. Because of the definition of the inductive topology $\mathfrak{B} \cap \boldsymbol{X}_t$ has to be a barrel in $\boldsymbol{X}_t$ in the $\boldsymbol{X}_t$-topology, for every $t>0$. Since $\boldsymbol{X}_t$ is a Hilbert space there exists an open neighbourhood of the origin Σ with $\Sigma \subset \mathfrak{B} \cap \boldsymbol{X}_t$. Again the conditions of Theorem 1.4.II are satisfied so that $\mathfrak{B}$ contains a set $\mathfrak{U}_{\psi,\varepsilon}$.

IV. We have to prove that every closed and bounded subset of $\boldsymbol{S}$ is compact iff for every $t>0$ the operator $e^{-t\mathscr{A}}$ is compact.

$\Leftarrow$) Suppose $e^{-t\mathscr{A}}$ is compact for every $t>0$. Let $\mathfrak{W}$ be a closed and bounded subset in $\boldsymbol{S}$. Then $\mathfrak{W} \subset \boldsymbol{X}_t$ for some $t>0$ and $\mathfrak{W}$ is closed and bounded in $\boldsymbol{X}_t$. See Theorem 1.6. Take τ, $0<\tau<t$. We claim that $\mathfrak{W}$ is a compact set in $\boldsymbol{X}_\tau$. To see this, consider the following commutative diagram where $\hookrightarrow$ denotes the natural injection:

$$\begin{tikzcd}
\boldsymbol{X}_t \arrow[r, hook] & \boldsymbol{X}_\tau \\
\boldsymbol{X}_0 \arrow[u, "e^{-t\mathscr{A}}"] \arrow[r, "e^{-(t-\tau)\mathscr{A}}"'] & \boldsymbol{X}_0 \arrow[u, "e^{-\tau\mathscr{A}}"']
\end{tikzcd}$$

The vertical arrows are isomorphisms. So $\hookrightarrow$ is a compact map and $\mathfrak{W}$ is compact in $\boldsymbol{X}_\tau$. Consider an open covering $\{C_\alpha\}$ of $\mathfrak{W}$ in $\boldsymbol{S}$, then $\{C_\alpha \cap \boldsymbol{X}_\tau\}$ is an open covering of $\mathfrak{W}$ in $\boldsymbol{X}_\tau$. Because of the compactness of $\mathfrak{W}$ in $\boldsymbol{X}_\tau$ there is a finite subcovering:

$$\mathfrak{W} \subset \bigcup_{i=1}^{N} (C_{\alpha_i} \cap \boldsymbol{X}_\tau).$$

But then certainly $\mathfrak{W} \subset \bigcup_{i=1}^{N} C_{\alpha_i}$, which shows the compactness of $\mathfrak{W}$.

$\Rightarrow$) Suppose $\boldsymbol{S}$ is Montel, i.e. each closed and bounded set is compact. Let $\{u_n\}$ be a bounded sequence in $\boldsymbol{X}$. Pick any fixed $t>0$. Consider the sequence $\{e^{-t\mathscr{A}}u_n\}$. Consider the closure in $\boldsymbol{S}$ of this sequence. This closure is a bounded set and, according to our assumption, compact. So $\{e^{-t\mathscr{A}}u_n\}$ contains a converging subsequence in $\boldsymbol{S}$: $e^{-t\mathscr{A}}u_{n_j} \to v$. This sequence certainly converges in $\boldsymbol{X}$. So $e^{-t\mathscr{A}}$ must be a compact operator.

V. $\Leftarrow$) Since $e^{-t\mathscr{A}}$ is HS and self-adjoint for all $t>0$ there exists an orthonormal basis $\{e_n\}$ of eigenvectors of $\mathscr{A}$ in $\boldsymbol{X}$. We order the eigenvalues with repetition according to multiplicity $0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots$. We have $\lambda_n \to \infty$ as $n \to \infty$. Thus we get the representation

$$e^{-t\mathscr{A}}u = \sum_{n=1}^{\infty} e^{-t\lambda_n}(u, e_n)e_n$$

where $\{e^{-t\lambda_n}\}$ is an l_2-sequence for each $t>0$ and hence an l_1-sequence. So $e^{-t\mathscr{A}}$ is a nuclear operator for each $t>0$.

$\boldsymbol{S}$ is a nuclear space iff for every semi-norm $\mathfrak{p}_\psi$, $\psi \in \mathbb{B}_+$, there is a semi-norm $\mathfrak{p}_\chi$, $\chi \in \mathbb{B}_+$, $\chi \ge \psi$, such that the canonical injection $\mathscr{J}: \hat{\boldsymbol{S}}_{\mathfrak{p}_\chi} \to \hat{\boldsymbol{S}}_{\mathfrak{p}_\psi}$ is nuclear. Cf. [Y]. Here the Banach space $\hat{\boldsymbol{S}}_{\mathfrak{p}}$ is the competion of $\boldsymbol{S}$ with respect to the norm $\mathfrak{p}$.

The canonical injection can be written

$$\mathscr{J}u = \sum_{j=1}^{\infty} \chi^{-1}(\lambda_j)\psi(\lambda_j)(\chi^2(\mathscr{A})u, \chi^{-1}(\mathscr{A})e_j)_0 \psi^{-1}(\lambda_j)e_j.$$

Note that $\{\chi^{-1}(\mathscr{A})e_j\}$ and $\{\psi^{-1}(\mathscr{A})e_j\}$ are orthogonal bases in the Hilbert spaces $\hat{\boldsymbol{S}}_{\mathfrak{p}_\chi}$ respectively $\hat{\boldsymbol{S}}_{\mathfrak{p}_\psi}$.

Now suppose there exists $\sigma \in \mathbb{B}_+$ such that $\sigma^{-1}(\mathscr{A})$ is a nuclear operator. If we take $\chi = \sigma\psi$ then the canonical injection can be written

$$\mathscr{J}u = \sum_{j=1}^{\infty} \sigma^{-1}(\lambda_j)(\chi^2(\mathscr{A})u, \chi^{-1}(\mathscr{A})e_j)_0 \psi^{-1}(\lambda_j)e_j.$$

Since $\{\sigma^{-1}(\lambda_j)\}$ is an l_1-sequence $\mathscr{J}$ is a nuclear operator.

It only remains to show that $\sigma \in \mathbb{B}_+$ with the desired properties can be constructed. To this end define integers $N(n)$, $n = 0, 1, 2, \ldots$, such that $N(0) = 0$, $N(n+1) > N(n)$, $\lambda_{N(n)+1} > \lambda_{N(n)}$ and

$$\sum_{j=N(n)}^{\infty} e^{-\lambda_j/n} < \frac{1}{n^2}.$$

Next define the sequence $\{v_k\}$ by

$$v_k = e^{-\lambda_k/n},\; N(n-1) < k \le N(n),\; n = 0, 1, 2, \ldots.$$

It is clear that $\sum_{k=1}^{\infty} \nu_k$ is convergent.

Define a step function $\sigma \in \mathbb{B}_+$ by setting $\sigma(\lambda) = e^{\lambda_k/n}$ on the interval

$$\left(\frac{\lambda_{k-1}+\lambda_k}{2}, \frac{\lambda_k+\lambda_{k+1}}{2}\right], \; k=2,3,\dots, \text{ and } \sigma(\lambda)=e^{\lambda_1} \text{ on } \left[0, \frac{\lambda_1+\lambda_2}{2}\right].$$

$\sigma^{-1}(\mathscr{A})$ is nuclear because it is self-adjoint, it has discrete spectrum, and the eigenvalues ν_k establish an l_1-sequence.

$\Rightarrow$) Suppose $\boldsymbol{S}$ is nuclear. There exists $\psi \in \mathbb{B}_+$ such that the injection $\hat{\boldsymbol{S}}_{\mathbf{P}_\psi} \to \boldsymbol{X}$ is a nuclear map. Therefore $\psi^{-1}(\mathscr{A})$ must be a nuclear operator and consequently also HS. But then $e^{-t\mathscr{A}} = \psi^{-1}(\mathscr{A})[\psi(\mathscr{A})e^{-t\mathscr{A}}]$ must be HS since the operator between [ ] is bounded for every $t>0$. □

CHAPTER 2. THE TRAJECTORY SPACE $\boldsymbol{T}_{\mathbf{X},\mathscr{A}}$

We now introduce our space of trajectories $\boldsymbol{T}_{\mathbf{X},\mathscr{A}}$. The elements in this space are candidates for becoming "generalized functions" (or "distributions").

DEFINITION 2.1. *$\boldsymbol{T}_{\mathbf{X},\mathscr{A}}$ denotes the complex vector space which consists of all mappings $F:(0,\infty)\to\boldsymbol{X}$ such that*

(i) F can be extended to an analytic function on the open right half-plane $\mathbb{C}^+$.

(ii) $\forall t, \tau \in \mathbb{C}^+ e^{-t\mathscr{A}}F(\tau) = F(t+\tau)$.

Such a mapping F will be called a trajectory. If for some $\xi>0$ $F_1(\xi)=F_2(\xi)$ then $F_1=F_2$. This follows immediately from the semi-group and analyticity properties. We shall use the notation $e^{-t\mathscr{A}}F$ for $F(t)$. It is immediate that $\forall t>0$ $e^{-t\mathscr{A}}F \in \boldsymbol{S}$. In chapters 2 and 3 $\boldsymbol{T}_{\mathbf{X},\mathscr{A}}$ is abbreviated by $\boldsymbol{T}$.

DEFINITION 2.2. *The embedding emb: $\boldsymbol{X}\to\boldsymbol{T}$ is defined by $(\text{emb } x)(t) = e^{-t\mathscr{A}}x$ for all $x\in\boldsymbol{X}$ and $t>0$.*

Sometimes we shall omit the symbol emb and loosely consider $\boldsymbol{X}$ as a subset of $\boldsymbol{T}$. Thus $\boldsymbol{S}\subset\boldsymbol{X}\subset\boldsymbol{T}$.

The question arises whether there exist trajectories $F\in\boldsymbol{T}$ such that $F(t)$ does not converge or, even $\|F(t)\|\uparrow\infty$ if $t\downarrow 0$. The answer is affirmative: simply take $x\in\boldsymbol{X}\setminus\mathscr{D}(\mathscr{A})$ and $F(t)=\mathscr{A}e^{-t\mathscr{A}}x$. More general examples are obtained by taking $\psi\in\mathbb{B}$, $u\in\boldsymbol{X}$ and defining $G(t)=\psi(\mathscr{A})e^{-t\mathscr{A}}u$, $t>0$. The next theorem shows that all elements of $\boldsymbol{T}$ arise in this way.

THEOREM 2.3. *For every $F\in\boldsymbol{T}$ there exists $w\in\boldsymbol{X}$ and $\psi\in\mathbb{B}^+$ such that $F(t)=\psi(\mathscr{A})e^{-t\mathscr{A}}w$ for every $t>0$. We write $F=\psi(\mathscr{A})w$.*

PROOF. Let $F\in\boldsymbol{T}$. Let $\mathscr{P}_n$ denote the projection $\int_{n-1}^{n} d\mathscr{E}_\lambda$. Then $e^{\mathscr{A}}\mathscr{P}_nF(1)\in\boldsymbol{X}$. Put $\alpha_n=\|e^{\mathscr{A}}\mathscr{P}_nF(1)\|$.

Let ψ be defined by

$$\psi(\lambda)=\begin{cases} \max(\alpha_n,1) & \text{for } \lambda\in(n-1,n],\ n\in\mathbb{N} \\ 1 & \text{elsewhere.}\end{cases}$$

We shall show that $\psi \in \mathbb{B}_+$. Let $t>0$. Then for $\lambda \in (n-1, n]$ with $\alpha_n \geq 1$ one estimates

$$|\psi(\lambda)e^{-\lambda t}|^2 \leq |\alpha_n e^{-(n-1)t}|^2 = e^{-2(n-1)t} \int_{n-1}^{n} e^{2\lambda} d(\mathscr{E}_\lambda F(1), F(1)) \leq$$

$$\leq e^{2t} \int_{n-1}^{n} d(\mathscr{E}_\lambda F(t), F(t)) \leq e^{2t} \|F(t)\|^2.$$

Take $w = \psi^{-1}(\mathscr{A})e^{\mathscr{A}}F(1)$. □

DEFINITION 2.4. *The strong topology in $\boldsymbol{T}$ is the topology induced by the semi-norms ϱ_n, $n \in \mathbb{N}$,*

$$\varrho_n(F) = \left\| F\left(\frac{1}{n}\right) \right\|_{\boldsymbol{X}}.$$

In other words a base of open 0-neighbourhoods is given by

$$\mathfrak{W}_{c_1 c_2 \ldots c_N} = \left\{ F \,\middle|\, \left\| F\left(\frac{1}{j}\right) \right\| < c_j, c_j > 0,\ 1 \leq j \leq N,\ N \in \mathbb{N} \right\}.$$

REMARK. The strong topology is equivalent to the topology of uniform convergence on compacta in $\mathbb{C}^+$.

THEOREM 2.5.

I. *$\boldsymbol{T}$ endowed with the strong topology is a Fréchet space, i.e. it is metrizable and complete.*

II. *A base of open sets $\{\sigma_{\mathfrak{U},t}\}$ is given by $\sigma_{\mathfrak{U},t} = \{F | F(t) \in \mathfrak{U}\}$, $\mathfrak{U}$ open in $\boldsymbol{X}$, $t>0$ and fixed.*
(In words: the set of trajections which pass at t through $\mathfrak{U}$.)
Each open set in $\boldsymbol{T}$ is a denumerable union of sets of this type.

PROOF. I. The topology is generated by a countable number of semi-norms. Hence $\boldsymbol{T}$ is metrizable. See [Y].

Further, since $\boldsymbol{T}$ is metrizable it is complete iff each Cauchy sequence converges. Let $\{F_m\}$ be a Cauchy sequence in $\boldsymbol{T}$.

Then, for each $n \in \mathbb{N}$, $\{F_m(1/n)\}$ is a Cauchy sequence in $\boldsymbol{X}$ which converges to an element $h_{1/n} \in \boldsymbol{X}$. For $\nu > n$ one has

$$F_m\left(\frac{1}{\nu}\right) = e^{((1/n)-(1/\nu))\mathscr{A}} F_m\left(\frac{1}{n}\right).$$

Since both sides converge and $e^{((1/n)-(1/\nu))\mathscr{A}}$ is a closed operator we have

$$h_{1/n} \in \boldsymbol{D}(e^{((1/n)-(1/\nu))\mathscr{A}}) \text{ and } h_{1/\nu} = e^{((1/n)-(1/\nu))\mathscr{A}} h_{1/n}.$$

Now we define

$$F(t) = e^{-(t-(1/\nu))\mathscr{A}} h_{1/\nu},\ \operatorname{Re} t \in \left[\frac{1}{\nu}, \frac{1}{\nu-1}\right),\ \nu \in \mathbb{N}.$$

Clearly $F \in \boldsymbol{T}$ and is the limit of $\{F_m\}$.

II. It is enough to prove that the semi-norms $F \mapsto \|F(t)\|$ are continuous. This follows simply from the estimate

$$\|F(t)\| \leq \|e^{-(t-(1/n))\mathscr{A}}\| \left\| F\left(\frac{1}{n}\right) \right\|$$

for $0<(1/n)<t$. □

THEOREM 2.6. *emb* $(\boldsymbol{S})$ *is everywhere dense in* $\boldsymbol{T}$.

PROOF. For any $F \in \boldsymbol{T}$ the sequence emb $(F(1/n))$ converges to F in the strong topology. □

THEOREM 2.7. *A set* $\mathfrak{B} \subset \boldsymbol{T}$ *is bounded iff for every* $t>0$ *the set* $\{F(t)|F \in \mathfrak{B}\}$ *is bounded in* $\boldsymbol{X}$.

PROOF. $\Rightarrow$) Each continuous semi-norm has to be bounded on $\mathfrak{B}$. Therefore $\varrho_n(F) = \|F(1/n)\|$ is a bounded function on $\mathfrak{B}$.

Because of the boundedness of $e^{-\tau\mathscr{A}}$ for each $\tau>0$ it then follows that $\{F(t)|F \in \mathfrak{B}\}$ is a bounded set for each fixed $t>0$.

$\Leftarrow$) Trivial. □

THEOREM 2.8. *A set* $\mathfrak{B} \subset \boldsymbol{T}$ *is bounded iff there exists a bounded set* $\mathfrak{V} \subset \boldsymbol{X}$ *and* $\psi \in \mathbb{B}_+$ *such that* $\mathfrak{B} = \psi(\mathscr{A})(\mathfrak{V})$.

PROOF. $\Leftarrow$) For each $t>0$ the set $\psi(\mathscr{A})e^{-t\mathscr{A}}(\mathfrak{V})$ is bounded in $\boldsymbol{X}$.

$\Rightarrow$) See [ETh], Ch. II cor. 2.5. It is not difficult to give an ad hoc proof in the spirit of Theorem 2.3. □

THEOREM 2.9. *A set* $\mathfrak{K} \subset \boldsymbol{T}$ *is compact iff for each* $t>0$ *the set* $\{F(t)|F \in \mathfrak{K}\}$ *is compact in* $\boldsymbol{X}$.

PROOF. $\Rightarrow$) If $\mathfrak{K}$ is compact then each sequence $\{F_n\} \subset \mathfrak{K}$ has a convergent subsequence. This means that in the set $\mathfrak{K}_t = \{F(t)|F \in \mathfrak{K}\}$, t fixed, each sequence has a convergent subsequence, which says that $\mathfrak{K}_t$ is compact in $\boldsymbol{X}$.

$\Leftarrow$) Let $\{F_n\}$ be a sequence in $\mathfrak{K}$. We must prove the existence of a converging subsequence. Consider the sequence $\{F_n(1)\} \subset \mathfrak{K}_1 \subset \boldsymbol{X}$. $\mathfrak{K}_1$ is compact, therefore a convergent subsequence in $\mathfrak{K}_1$ exists. Denote it by $\{F_n^1(1)\}$. The sequence $\{F_n^1(\frac{1}{2})\}$ has a convergent subsequence in $\mathfrak{K}_{\frac{1}{2}}$. Denote it by $\{F_n^2(\frac{1}{2})\}$.

Proceeding in this way we arrive at sequences $\{F_n^m\} \subset \mathfrak{K}$ such that $\{F_n^m\} \subset \{F_n^l\}$ for $m>l$ and $\{F_n^m(1/m)\}$ converges to an element in $\mathfrak{K}_{1/m}$. The "diagonal sequence" $\{F_n^n\}$ has the property that $\{F_n^n(t)\}$ converges to $F(t) \in \mathfrak{K}_t$. But then also $F_n^n \to F$ in the strong topology. □

THEOREM 2.10. *A set* $\mathfrak{K} \subset \boldsymbol{T}$ *is compact iff there exists a compact set* $\mathfrak{W} \subset \boldsymbol{X}$ *and* $\psi \in \mathbb{B}_+$ *such that* $\mathfrak{K} = \psi(\mathscr{A})(\mathfrak{W})$.

PROOF. $\Leftarrow$) For each $t>0$ the operator $\psi(\mathscr{A})e^{-t\mathscr{A}}$ is bounded. Therefore $\psi(\mathscr{A})e^{-t\mathscr{A}}(\mathfrak{W})$ is a compact set in $\boldsymbol{X}$.

$\Rightarrow$) See [ETh], Ch. II cor. 2.13.

In the last theorem of this chapter we describe the properties of our topological vector space $\boldsymbol{T}$ in the standard terminology of topological vector spaces [SCH].

THEOREM 2.11.

I. $\boldsymbol{T}_{\mathrm{X},\mathscr{A}}$ *is bornological.*

II. $\boldsymbol{T}_{\mathrm{X},\mathscr{A}}$ *is barreled.*

III. $\boldsymbol{T}_{\mathrm{X},\mathscr{A}}$ *is Montel iff for every* $t>0$ *the operator* $e^{-t\mathscr{A}}$ *is compact as an operator on* $\boldsymbol{X}$.

IV. $\boldsymbol{T}_{\mathrm{X},\mathscr{A}}$ *is nuclear iff for every* $t>0$ *the operator* $e^{-t\mathscr{A}}$ *is a HS operator on X.*

PROOF. I, II. $\boldsymbol{T}$ is bornological and barreled because it is metrizable. For a simple proof see [SCH] II.8.

III. $\Rightarrow$) Suppose $\boldsymbol{T}$ is Montel. Take a bounded sequence $\{x_n\}\subset\boldsymbol{X}$. The sequence $\{F_n\}$ defined by $F_n(t)=e^{-t\mathscr{A}}x_n$ is a bounded set in $\boldsymbol{T}$. Because of our assumption the closure of this set is compact. But then, by theorem 2.9, for each $t>0$ the sequence $F_n(t)$ contains a converging subsequence. This shows the compactness of $e^{-t\mathscr{A}}$.

$\Leftarrow$) Suppose $e^{-t\mathscr{A}}$ is compact for every $t>0$. Let $\mathfrak{B}$ be a closed and bounded set. Then the set $\mathfrak{B}_t=\{F(t)|F\in\mathfrak{B},\ t>0$ and fixed$\}$ is bounded. Since $\mathfrak{B}_{t+\tau}=e^{-t\mathscr{A}}(\mathfrak{B}_t)$ it follows that $\mathfrak{B}_{t+\tau}$ is precompact in $\boldsymbol{X}$. Now take any sequence $\{G_n\}\subset\mathfrak{B}$. Because of the precompactness of each $\mathfrak{B}_t$ the "diagonal procedure" of the proof of theorem 2.9 yields a converging subsequence of $\{G_n\}$. This subsequence converges to an element in $\mathfrak{B}$ because $\mathfrak{B}$ is closed. We conclude that $\mathfrak{B}$ is compact.

IV. $\Rightarrow$) Suppose $\boldsymbol{T}$ is a nuclear space. Take $n\in\mathbb{N}$. There exists a seminorm $\varrho_m\geq\varrho_n$ such that the natural injection $\hat{\boldsymbol{T}}_{\varrho_m}\to\hat{\boldsymbol{T}}_{\varrho_n}$ is nuclear.

This natural injection is realized by the map $e^{-(1/n-1/m)\mathscr{A}}$ which must therefore be nuclear. It follows that $e^{-t\mathscr{A}}$ is HS for each $t>0$.

$\Leftarrow$) Suppose for each $t>0$ $e^{-t\mathscr{A}}$ is HS. Because of the semi-group property $e^{-t\mathscr{A}}$ is also nuclear. Let $m>n$. The natural injection $\hat{\boldsymbol{T}}_{\varrho_m}\to\hat{\boldsymbol{T}}_{\varrho_n}$ is realized by $e^{-(1/n-1/m)\mathscr{A}}$ and is therefore nuclear. This is enough for $\boldsymbol{T}$ to be nuclear. □

CHAPTER 3. THE PAIRING OF $\boldsymbol{S}_{\mathrm{X},\mathscr{A}}$ AND $\boldsymbol{T}_{\mathrm{X},\mathscr{A}}$

We now consider a pairing of $\boldsymbol{S}$ and $\boldsymbol{T}$. It turns out that $\boldsymbol{S}$ and $\boldsymbol{T}$ can be considered as each others strong duals.

DEFINITION 3.1. *On* $\boldsymbol{S}_{\mathrm{X},\mathscr{A}}\times\boldsymbol{T}_{\mathrm{X},\mathscr{A}}$ *we introduce a sesquilinear form by*

$$\langle u,F\rangle_{\boldsymbol{X}}=(e^{t\mathscr{A}}u,F(t))_{\boldsymbol{X}}.$$

Note that this definition makes sense for $t>0$ sufficiently small. Note also that because of the semi-group property and self-adjointness of $e^{-t\mathscr{A}}$ the definition does not depend on the choice of t. We remark that $\langle u_0, F\rangle = 0$ for all $F \in \boldsymbol{T}$ implies $u_0 = 0$ and $\langle u, F_0\rangle = 0$ for all $u \in \boldsymbol{S}$ implies $F_0 = 0$. These two facts easily follow by taking $F = \text{emb}\ (u_0)$, respectively the denseness of emb $(\boldsymbol{S})$ in $\boldsymbol{T}$.

THEOREM 3.2.

I. *For each $F \in \boldsymbol{T}$ the linear functional $\langle u, F\rangle$, which maps $\boldsymbol{S}$ onto $\mathbb{C}$, is continuous in the strong topology of $\boldsymbol{S}$.*

II. *For each strongly continuous linear functional l on $\boldsymbol{S}$ there exists $G \in \boldsymbol{T}$ such that $l(u) = \langle u, G\rangle$ for all $u \in \boldsymbol{S}$.*

III. *For each $v \in \boldsymbol{S}$ the linear functional $\overline{\langle v, G\rangle}$, which maps $\boldsymbol{T}$ onto $\mathbb{C}$, is continuous in the strong topology of $\boldsymbol{T}$.*

IV. *For each strongly continuous linear functional m on $\boldsymbol{T}$ there exists $w \in \boldsymbol{S}$ such that $m(F) = \overline{\langle v, F\rangle}$ for all $F \in \boldsymbol{T}$.*

PROOF. I. The function $u \mapsto \langle u, F\rangle$ is continuous on $\boldsymbol{S}$ iff for all $t > 0$ it is continuous in the $\boldsymbol{X}_t$-norm when restricted to this subspace. Indeed

$$|\langle u, F\rangle| = |(e^{t\mathscr{A}}u, F(t))| \leq \|e^{t\mathscr{A}}u\|\ \|F(t)\| = \|F(t)\|\ \|u\|_t.$$

II. For fixed $t > 0$ the mapping $e^{-t\mathscr{A}}: \boldsymbol{X} \to \boldsymbol{S}$ is continuous. Therefore the mapping $l(e^{-t\mathscr{A}}x): \boldsymbol{X} \to \mathbb{C}$ is continuous. From Riess' theorem follows the existence of $b_t \in \boldsymbol{X}$ such that $l(e^{-t\mathscr{A}}x) = (x, b_t)$ for all $x \in \boldsymbol{X}$. If we replace x by $e^{-t\mathscr{A}}y$, $y \in \boldsymbol{X}$, it follows from the self-adjointness of $e^{-t\mathscr{A}}$ that $b_{t+\tau} = e^{-\tau\mathscr{A}}b_t$. We take G such that $G(t) = b_t$. Then $l(u) = \langle u, G\rangle$ for all $u \in \boldsymbol{S}$.

III. Since $\boldsymbol{T}$ is metrizable it is sufficient to prove continuity for sequences in $\boldsymbol{T}$. Let $G_n \to 0$ in the strong topology. Then for sufficiently small t we have $\overline{\langle v, G_n\rangle} = \overline{(e^{t\mathscr{A}}v, G_n(t))} \to 0$ because $G_n(t) \to 0$ in $\boldsymbol{X}$-sense.

IV. Take $\psi \in \mathbb{B}_+$. Take a sequence $\{x_n\} \subset \boldsymbol{X}$, $x_n \to 0$ in $\boldsymbol{X}$. Define $\psi(\mathscr{A})x_n \in \boldsymbol{T}$ by $(\psi(\mathscr{A})x_n)(t) = \psi(\mathscr{A})e^{-t\mathscr{A}}x_n$. Since $\psi(\mathscr{A})e^{-t\mathscr{A}}$ is a bounded operator $\psi(\mathscr{A})x_n \to 0$ strongly in $\boldsymbol{T}$. By taking $\psi = 1$ we conclude first that the restriction of m to $\boldsymbol{X}$ has the Riess representation $m(x) = (x, \theta)$ for some $\theta \in \boldsymbol{X}$. Secondly we conclude that $m(\psi(\mathscr{A})x)$ is a continuous function on $\boldsymbol{X}$ for each $\psi \in \mathbb{B}_+$. Using the self-adjointness of $\psi(\mathscr{A})$ it follows that $\theta \in \mathscr{D}(\psi(\mathscr{A}))$ for each $\psi \in \mathbb{B}_+$. But then, by Theorem 1.10, $\theta \in \boldsymbol{S}$. Finally, as $\boldsymbol{X}$ is dense in $\boldsymbol{T}$ the representation $m(F) = (F, \theta) = \overline{\langle \theta, F\rangle}$ is valid for all $F \in \boldsymbol{T}$. □

DEFINITION 3.3. *The weak topology on $\boldsymbol{S}$ is the topology induced by the semi-norms $\mathfrak{p}_F(u) = |\langle u, F\rangle|$, $F \in \boldsymbol{T}$.*

The weak topology on $\boldsymbol{T}$ is the topology indiced by the semi-norms $\varrho_v(G) = |\langle v, G\rangle|$, $v \in \boldsymbol{S}$.

A simple standard argument, [CH] II § 22, shows that the weakly continuous functionals on $\boldsymbol{S}$ are all obtained by pairing with elements of $\boldsymbol{T}$ and vice versa. Together with Theorem 3.2 it follows then that $\boldsymbol{S}$ and $\boldsymbol{T}$ are reflexive both in the strong and the weak topology.

In the next theorem we show that in both spaces $\boldsymbol{S}$ and $\boldsymbol{T}$ weakly bounded sets are strongly bounded.

THEOREM 3.4 (Banach-Steinhaus).

I. *Let $\Xi \subset \boldsymbol{T}$ be such that for each $g \in \boldsymbol{S}$ there exists $M_g > 0$ such that for every $F \in \Xi$ one has $|\langle g, F\rangle| \leq M_g$, then for each $t > 0$ there exists $C_t > 0$ such that for every $F \in \Xi$ one has $\|F(t)\| \leq C_t$. So weakly bounded sets in $\boldsymbol{T}$ are strongly bounded.*

II. *Let $\Theta \subset \boldsymbol{S}$ be such that for every $F \in \boldsymbol{T}$ there exists $M_F > 0$ such that for every $f \in \Theta$ one has $|\langle f, F\rangle| \leq M_F$, then there exists $\tau > 0$ and $c > 0$ such that $\Theta \subset \boldsymbol{X}_\tau$ and $\|f\|_\tau < c$ for all $f \in \Theta$. So weakly bounded sets in $\boldsymbol{S}$ are strongly bounded.*

PROOF. I. From the assumption it follows that for each $h \in \boldsymbol{X}$ and each $t > 0$ there exists a constant $M_{h,t} > 0$ such that for every $F \in \Xi$, $|\langle e^{-t\mathscr{A}}h, F\rangle| \leq M_{h,t}$ or $|(h, F(t))| \leq M_{h,t}$. From the Banach-Steinhaus theorem in Hilbert space it then follows that the set $\{F(t) | F \in \Xi\}$, t fixed, is a bounded set in $\boldsymbol{X}$.

II. Let $\psi \in \mathbb{B}_+$. Let $w \in \mathscr{D}(\psi(\mathscr{A}))$. Then $\psi(\mathscr{A})^2 w$, defined by $(\psi(\mathscr{A})^2 w)(t) = \psi(\mathscr{A})^2 e^{-t\mathscr{A}} w$ belongs to $\boldsymbol{T}$. From our assumption it follows that for every $f \in \Theta$

$$|\langle f, \psi(\mathscr{A})^2 w\rangle| = |(\psi(\mathscr{A})f, \psi(\mathscr{A})w)| < M_{\psi, w}.$$

From the Banach-Steinhaus theorem in Hilbert space it then follows that Θ is a bounded set in the Hilbert space $\boldsymbol{X}_\psi$, i.e. the completion of $\boldsymbol{S}$ with respect to the norm $\|\psi(\mathscr{A})\cdot\|$. But this means that Θ is bounded in $\boldsymbol{S}$, since each seminorm $\mathfrak{p}_\psi$, $\psi \in \mathbb{B}_+$, is bounded on it. □

In the next two theorems we give characterizations of weak convergence of sequences both in $\boldsymbol{S}$ and $\boldsymbol{T}$.

THEOREM 3.5. *$u_n \to 0$ in the weak topology of $\boldsymbol{S}$ iff*

$$\exists t > 0 \ \{u_n\} \subset \boldsymbol{X}_t \text{ and } \forall w \in \boldsymbol{X}_t \ (w, u_n)_t \to 0.$$

PROOF. Weak convergence of $\{u_n\}$ in $\boldsymbol{X}_t$ means weak convergence of $e^{t\mathscr{A}}u_n$ in $\boldsymbol{X}$.

$\Leftarrow$) Let $F \in \boldsymbol{T}$. $\langle u_n, F\rangle = (e^{t\mathscr{A}}u_n, F(t))$. Since $e^{t\mathscr{A}}u_n \to 0$ weakly in $\boldsymbol{X}$ and $F(t) \in \boldsymbol{X}$, it follows that $\langle u_n, F\rangle \to 0$.

$\Rightarrow$) First we remark that weak convergence in $\boldsymbol{S}$ implies weak convergence in $\boldsymbol{X}$. Therefore for any $w \in \boldsymbol{X}$ any $L > 0$, $t > 0$,

$$\int_0^L e^{\lambda t} d(\mathscr{E}_\lambda u_n, w) \to 0 \text{ as } n \to \infty.$$

$\{u_n\}$ is a bounded set in $\boldsymbol{S}$. Therefore $\exists \tau > 0 \ \exists \theta > 0 \ \forall n \in \mathbb{N} \ \|u_n\|_\tau \leq \theta$. We shall prove that $u_n \to 0$ weakly in $\boldsymbol{X}_\tau$. Denote the projection operator $\int_L^\infty d\mathscr{E}_\lambda$ by Π_L,

Π_L commutes with $e^{\tau\mathscr{A}}$. Taken $w \in \boldsymbol{X}$, then

$$(*) \qquad (e^{\tau\mathscr{A}}u_n, w) = \int_0^L e^{\lambda t} d(\mathscr{E}_\lambda u_n, w) + (\Pi_L e^{\tau\mathscr{A}}u_n, \Pi_L w).$$

We have $\|\Pi_L e^{\tau\mathscr{A}}u_n\| \leq \theta$ and $\|\Pi_L w\| \to 0$ as $L \to \infty$. Therefore if we take L large enough the second term in (*) is smaller than $\frac{1}{2}\varepsilon$ uniformly for all n. As we have just seen the first term in (*) can be made smaller than $\frac{1}{2}\varepsilon$ by taking n large enough. This finishes the proof. □

COROLLARY 3.6.
I. *Strong convergence of a sequence in* $\boldsymbol{S}$ *implies its weak convergence.*
II. *Any bounded sequence in* $\boldsymbol{S}$ *has a weakly converging subsequence.*

THEOREM 3.7. $F_n \to 0$ *in the weak topology of* $\boldsymbol{T}$ *iff*

$$\forall t > 0 \ \forall v \in \boldsymbol{X} \ (v, F_n(t))_0 \to 0.$$

PROOF. $\Rightarrow$) For any $v \in \boldsymbol{X}$, $\langle e^{-t\mathscr{A}}v, F_n \rangle = (v, F_n(t))_0 \to 0$ as $n \to \infty$.
$\Leftarrow$) For any $\varphi \in \boldsymbol{S}$ and t sufficiently small

$$\langle \varphi, F_n \rangle = (e^{t\mathscr{A}}\varphi, F_n(t)) \to 0$$

because $e^{t\mathscr{A}}\varphi \in \boldsymbol{X}$. □

COROLLARY 3.8.
I. *Strong convergence of a sequence in* $\boldsymbol{T}$ *implies its weak convergence.*
II. *Any bounded sequence in* $\boldsymbol{T}$ *has a weakly convergent subsequence. Cf. Theorem* 2.9.

It looks reasonable to conjecture that the weak topology on $\boldsymbol{S}$ is the inductive limit topology with respect to the Hilbert spaces $\boldsymbol{X}_t$, now endowed with the weak topology. It is easily seen that the weak topology on $\boldsymbol{T}$ is induced by the semi-norms $\varrho_{t,v}$, $t>0$, $v \in \boldsymbol{X}$, and $\varrho_{t,v}(F) = |(v, F(t))|$. We will not pursue these things further here. The next theorem deals with the question: When does weak convergence of a sequence imply its strong convergence?

THEOREM 3.9. *The following three statements are equivalent.*
I. *For each* $t>0$, $e^{-t\mathscr{A}}$ *is a compact operator on* $\boldsymbol{X}$.
II. *Each weakly convergent sequence in* $\boldsymbol{S}$ *converges strongly in* $\boldsymbol{S}$.
III. *Each weakly convergent sequence in* $\boldsymbol{T}$ *converges strongly in* $\boldsymbol{T}$.

PROOF. I$\Rightarrow$II. A weakly convergent sequence in $\boldsymbol{S}$ converges, for some $t>0$, weakly in $\boldsymbol{X}_t$. See Theorem 3.5. Because of the assumption the natural injection $\boldsymbol{X}_t \subset \boldsymbol{X}_\alpha$, $0<\alpha<t$, is compact. Cf. the proof of Theorem 1.11. But then our sequence converges strongly in $\boldsymbol{X}_\alpha$.

II$\Rightarrow$I. Take any sequence $\{f_n\} \subset \boldsymbol{X}$, $f_n \to 0$ weakly in $\boldsymbol{X}$. For each $t>0$ and each $F \in \boldsymbol{T}$ we have $\langle e^{-t\mathscr{A}}f_n, F \rangle \to 0$. Thus $e^{-t\mathscr{A}}f_n \to 0$ weakly in $\boldsymbol{S}$. Because of the assumption $e^{-t\mathscr{A}}f_n \to 0$ strongly in $\boldsymbol{S}$. And so $\|e^{-t\mathscr{A}}f_n\| \to 0$. This shows that $e^{-t\mathscr{A}}$ must be compact.

I$\Rightarrow$III. Let $\{F_n\} \subset \boldsymbol{T}$. Suppose $\forall g \in \boldsymbol{S}\ \langle g, F_n \rangle \to 0$. Then

$$\forall h \in \boldsymbol{X}\ \forall \alpha > 0\ \langle e^{-\alpha\mathscr{A}}h, F_n \rangle = (h, F_n(\alpha)) \to 0.$$

This means that $\forall \alpha > 0\ F_n(\alpha) \to 0$ weakly in $\boldsymbol{X}$. Using the compactness of $e^{-\beta\mathscr{A}}$, $\beta > 0$, we find that $F_n(\alpha+\beta) = e^{-\beta\mathscr{A}}F_n(\alpha) \to 0$ strongly in $\boldsymbol{X}$. Therefore $\forall t > 0$ $\|F_n(t)\| \to 0$.

III$\Rightarrow$I. Let $\{v_n\} \subset \boldsymbol{X}$ be such that $v_n \to 0$ weakly in $\boldsymbol{X}$. Then $v_n \to 0$ weakly in $\boldsymbol{T}$ because $\forall g \in \boldsymbol{S}\ \langle g, v_n \rangle = (g, v_n) \to 0$. Now the assumption says $v_n \to 0$ strongly in $\boldsymbol{T}$. This means $\forall t > 0\ e^{-t\mathscr{A}}v_n \to 0$ strongly in $\boldsymbol{X}$. The compactness of $e^{-t\mathscr{A}}$ follows for each $t > 0$. □

One might wonder whether in the case that $e^{-t\mathscr{A}}$ is compact for each $t > 0$ the strong and weak topologies coincide. This however cannot be true since a set which belongs to a weak base of 0-neighbourhoods always contains a closed subspace of finite codimension. Generally speaking such "large" sets do not fit in a strongly open 0-neighbourhood.

It is relatively simple to show that both in $\boldsymbol{S}$ and $\boldsymbol{T}$ the weakly compact sets are just the (weakly) closed and bounded sets. Then it is not difficult to prove that $\boldsymbol{S}$ and $\boldsymbol{T}$ with their strong topologies are *Mackey spaces*. That is the strong topology is the finest locally convex topology for which the dual of $\boldsymbol{S}$ (respectively $\boldsymbol{T}$) is $\boldsymbol{T}$ (respectively $\boldsymbol{S}$). (All spaces which are either barreled or bornological are Mackey. See [SCH] IV.4.)

ACKNOWLEDGEMENT

The author thanks Dr. S.J.L. van Eijndhoven for his contribution to the revision of the manuscript.

REFERENCES

See Part A.